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A Remark on Different Norms and Analyticity for Many-Particle Interactions

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We compare a recent result of Dobrushin and Martirosyan with previous results by Gallavotti and Miracle-Sole and by Israel and point out that the analytic behavior at high temperatures for many-particle interactions is different depending on whether the interactions are weighted with a lattice-gas or Ising norm or, on the other hand, with the supremum norm.

KEY WORDS: Many-particle interactions; physically equivalent interactions; inequivalent norms; analyticity.

In a recent paper, Dobrushin and Martirosyan[1] proved that at high temperature, lattice systems can be analytic in spaces containing many-particle interactions *only* if the many-particle terms are exponentially suppressed. At first sight this result seems to contradict earlier analyticity results[3-5] which do not need such an exponential factor. Here we clarify this apparent contradiction by pointing out that these results use different, inequivalent, norms.

For general classical lattice systems with configuration space $S^{\mathbb{Z}^d}$, an interaction is a family of functions $\Phi = \{\Phi_X\}$, where X runs over the finite subsets of the lattice $\mathbb{Z}^d$ and each function $\Phi_X: S^X \to \mathbb{C}$ is continuous (here the space S is assumed to have some natural topology and $S^{\mathbb{Z}^d}$ is equipped with the product topology). This description contains a great deal of redundancy because many interactions describe the same physics. Such interactions, which have Hamiltonians differing at most in a constant and a boundary term, are called physically equivalent. They define the same equilibrium states. One can factor out this physical equivalence by identifying all equivalent interactions and looking only at the equivalence classes.[16,17] However, such a quotient space is a little abstract and not

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easy to visualize or to handle. An alternative solution is to try to find a linear "section," that is, a subspace of the vector space of all interactions containing exactly one representative of each class, chosen according to a well-defined prescription.

If the one-particle space S has only two elements—we will consider $S=\{-1,1\}$—there are two popular choices of such sections, which are generally considered special cases, in both of which the interactions are defined by functions $J^{\text{spin}}(\cdot)$ ("coupling constants") on the finite subsets of the lattice: the "Ising spin language," in which $\Phi_X^{\text{spin}}(\sigma)=J^{\text{spin}}(X)\sigma^X$; and the "lattice-gas language," where $\Phi_X^{\text{gas}}(\sigma)=J^{\text{gas}}(X)((\sigma+1)/2)^X$. [We adhere to the standard notation $(a+b\sigma)^X=\prod_{i\in X}(a+b\sigma_i)$ for constants a and b and each $\sigma_i\in\{-1,1\}$.]

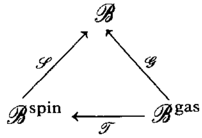
For finite-range interactions—that is, when the Φ_X are zero if the diameter of X exceeds a certain value—there is exactly one Φ^{spin} or Φ^{gas} for each class of physically equivalent interactions. Hence it is equivalent to work with the big general space of interactions or with the more economical spin or lattice-gas formulations. This equivalence is no longer true if many-particle interactions are included. However, it was somehow tacitly assumed (or at least the possibility of this not being so was never emphasized) that the main thermodynamic properties of an interaction did not depend on the type of "language." We point out here that the recent result of ref. 1 implies that for many-particle models the existence of one of these major properties—analyticity at high temperature or low density—strongly depends on the space of interactions considered.

To be more precise, let us denote by $\mathscr{B}$, $\mathscr{B}^{\text{spin}}$, and $\mathscr{B}^{\text{gas}}$ the vector spaces of general interactions, spin interactions, and lattice-gas interactions, respectively, and let $\mathscr{S}$ and $\mathscr{G}$ be the canonical injections of $\mathscr{B}^{\text{spin}}$ and $\mathscr{B}^{\text{gas}}$ into $\mathscr{B}$:

$$(\mathscr{S}\Phi^{\text{spin}})_X(\sigma)=J^{\text{spin}}(X)\sigma^X$$

$$(\mathscr{G}\Phi^{\text{gas}})_X(\sigma)=J^{\text{gas}}(X)\left(\frac{\sigma+1}{2}\right)^X$$

This defines a natural "translation" between spin and lattice-gas languages, namely the map $\mathscr{T}$ that makes commutative the diagram



This map is defined[6] by transforming the coupling constants in the form[2]

$$(\mathscr{T} J^{\text{gas}})(X) = \sum_{Y \supset X} 2^{-|Y|} J^{\text{gas}}(Y)$$

($|\cdot|$ stands for "cardinal of"). (This map is defined, for instance, for finite-range interactions, or in the spaces $\mathscr{B}_g^{\text{gas}}$ defined below, if $g \geqslant 1$.)

Restricted to finite-range interactions, both $\mathscr{S}$ and $\mathscr{G}$ have left inverses π_{spin} and π_{gas} and $\mathscr{T}$ has an inverse $\mathscr{T}^{-1}$, defined as follows. The projection π_{spin} is defined by the spin coupling constants[6]

$$(\pi_{\text{spin}} \Phi)(X) = \sum_{Y \supset X} [\operatorname{Tr} \Phi_Y(\sigma) \sigma^X] \tag{1}$$

where $\operatorname{Tr} = \prod_i [\frac{1}{2} \sum_{\sigma_i = \pm 1}]$. The map $\mathscr{T}^{-1}$ is defined by the lattice-gas coupling constants[6]

$$(\mathscr{T}^{-1} J^{\text{spin}})(X) = 2^{|X|} \sum_{Y \supset X} (-1)^{|Y \setminus X|} J^{\text{spin}}(Y) \tag{2}$$

Finally, the projection π_{gas} follows from the composition of these two maps:

$$\begin{aligned} (\pi_{\text{gas}} \Phi)(X) &= (\mathscr{T}^{-1} \pi_{\text{spin}} \Phi)(X) \\ &= 2^{|X|} \sum_{Y \supset X} \operatorname{Tr}[\Phi_Y(\sigma)\, \sigma^X (1-\sigma)^{Y \setminus X}] \end{aligned} \tag{3}$$

If the interactions are allowed to involve arbitrarily many particles, norms must be introduced to handle convergence problems. Several norms are usually considered, characterized by a function $g: \mathbb{Z}^+ \to \mathbb{R}^+ \setminus \{0\}$. Let us, for simplicity, concentrate on translation-invariant interactions. For general interactions, let $\mathscr{B}_g$ denote the Banach space of interactions for which the norm $\|\Phi\|_g = \sum_{X \ni 0} \|\Phi_X\|_\infty \, g(|X|)$ is finite. For the spin and lattice-gas formulations, Banach spaces $\mathscr{B}_g^{\text{spin}}$ and $\mathscr{B}_g^{\text{gas}}$ are defined through the norms $\|\Phi^{\text{spin}}\|_g^{\text{spin}} = \sum_{X \ni 0} |J^{\text{spin}}(X)| \, g(|X|)$ and its analog for the lattice gas, so the inclusions $\mathscr{S}$ and $\mathscr{G}$ become isometries. The corresponding spaces of finite-range interactions—denoted $\mathscr{B}_{0g}$, $\mathscr{B}_{0g}^{\text{spin}}$, and $\mathscr{B}_{0g}^{\text{gas}}$, respectively—are dense subspaces. The map $\mathscr{T}$ becomes a bounded map from $\mathscr{B}_{0g}^{\text{gas}}$ to $\mathscr{B}_{0g}^{\text{spin}}$ for the usual choices of g: $\|\mathscr{T}\|_{g=1} = 1/2$, $\|\mathscr{T}\|_g \leqslant e^\lambda/(1+e^\lambda)$ for $g(t) = e^{\lambda t}$ with $\lambda \geqslant 0$—in general $\|\mathscr{T}\|_g \leqslant 1/2$ for g increasing—and $\|\mathscr{T}\|_g \leqslant 1$ for $g(t) = 1/t$.[6]

[2] Note that we use the same symbol for a map between interactions and for the corresponding map between coupling constants.

The inverse maps π_{spin}, π_{gas}, and $\mathscr{T}^{-1}$ make the diagram

$$\begin{array}{ccc} & \mathscr{B}_{0g} & \\ {}^{\pi_{\text{spin}}}\swarrow & & \searrow^{\pi_{\text{gas}}} \\ \mathscr{B}_{0g}^{\text{spin}} & \xrightarrow[\mathscr{T}^{-1}]{} & \mathscr{B}_{0g}^{\text{gas}} \end{array} \tag{4}$$

commutative, but, as shown in ref. 6 and in the examples below, they are *unbounded*—at least for the weights g of interest here. Therefore, even when the formulas (1) and (3) make sense for $g \geq 1$, the projections π_{spin} and π_{gas} cannot be extended continuously to the whole $\mathscr{B}_g$; in fact, as shown below, they map a dense set of interactions in $\mathscr{B}_g$ outside of $\mathscr{B}_g^{\text{spin}}$ and $\mathscr{B}_g^{\text{gas}}$ respectively. In other words, for many-particle interactions the spaces of lattice-gas and spin interactions are not sections of $\mathscr{B}_g$. They do not contain representatives of all the classes of physically equivalent interactions, and for those classes represented, the selected representative may have an extremely large norm. A natural question that arises is whether there are better choices for linear sections (or a choice at all). The obvious thing to do would be to take for each class the representative with minimum norm (which exists by completeness), but on the one hand such interactions could not be defined algorithmically as the spin or lattice-gas ones, and on the other hand, the section so obtained would not be a subspace. (A ferromagnetic and an antiferromagnetic spin interaction are both of minimal norm, but their sum in general is not so, due to frustration.) Likewise, the function $\mathscr{T}^{-1}$ defined in (2) makes sense for $g \geq 1$, but maps some interactions on $\mathscr{B}_g^{\text{spin}}$ (in fact, a dense set) outside $\mathscr{B}_g^{\text{gas}}$. This means that $\mathscr{B}_g^{\text{spin}}$ is closer to being a section in the sense that it contains representatives of more classes of equivalent interactions than $\mathscr{B}_g^{\text{gas}}$.

To sum up the previous discussion, let us denote $[\mathscr{B}_g]$ the quotient space modulo physical equivalence, that is, the Banach space of classes of equivalent interactions with norm equal to the minimal norm within each class. Then all the above maps are well defined if $\mathscr{B}_g$ is replaced by $[\mathscr{B}_g]$, and the unboundedness of the inverse maps implies, with obvious identifications,

$$[\mathscr{B}_g] \supsetneqq \mathscr{B}_g^{\text{spin}} \supsetneqq \mathscr{B}_g^{\text{gas}}$$

For completeness, let us present here examples showing the unboundedness of the inverse maps. Examples for $\mathscr{T}^{-1}$ are well known[6]; in particular, if D_n denotes the cube $[0, n-1]^d \cap \mathbb{Z}^d$, the spin interactions defined by

$$J_n^{\text{spin}}(Y) = \begin{cases} \dfrac{(-1)^{|D_n|}}{|D_n|\; g(|D_n|)} & \text{if } \; Y \text{ is a translate of } D_n \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

have norm

$$\|\Phi_n^{\text{spin}}\|_g^{\text{spin}} = 1 \tag{6}$$

and, on the other hand,

$$(\mathscr{T}^{-1} J_n^{\text{spin}})(X) = \frac{2^{|X|}(-1)^{|X|}}{|D_n|\ g(|D_n|)}\ |\{s: D_n + s \supset X\}|$$

Hence

$$\begin{aligned}
\|\mathscr{T}^{-1}\Phi_n^{\text{spin}}\|_g^{\text{gas}} &= \frac{1}{|D_n|\ g(|D_n|)} \sum_{X \ni 0} 2^{|X|} g(|X|)\ |\{s: D_n + s \supset X\}| \\
&\geqslant \frac{1}{|D_n|\ g(|D_n|)} \sum_{\substack{X \ni 0 \\ X \subset D_n}} 2^{|X|} g(|X|) \\
&= \frac{1}{|D_n|\ g(|D_n|)} \sum_{l=1}^{|D_n|} \binom{|D_n| - 1}{l-1} 2^l g(l) \\
&\geqslant \left\{ \begin{array}{ll} \dfrac{2e^{\lambda}(1+2e^{\lambda})^{|D_n|-1}}{|D_n|} & \text{if} \quad g(t) = e^{\lambda t} \\ \dfrac{3^{|D_n|} - 1}{|D_n|} & \text{if} \quad g(t) = \dfrac{1}{t} \end{array} \right\} \xrightarrow[n \to \infty]{} \infty
\end{aligned} \tag{7}$$

which shows that $\mathscr{T}^{-1}$ is unbounded.

Regarding the unboundedness of π_{spin}, there is a probabilistic argument in ref. 6 for $g(t) = 1/t$ that can be extended almost verbatim to $g(t) = e^{\lambda t}$ for $\lambda \geqslant 0$ small enough. Let us give here a concrete example for $g = 1$. Consider the (complex) potentials $\Phi_n \in \mathscr{B}_0$ defined by

$$(\Phi_n)_Y(\sigma) = \begin{cases} \dfrac{1}{|D_n|} \exp\left[\dfrac{i}{|D_n|^{1/2}} \sum_{j \in Y} \sigma_j\right] & Y \text{ is a translate of } D_n \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

which have $g = 1$ norm

$$\|\Phi_n\|_{g=1} = 1 \tag{9}$$

On the other hand, the corresponding projections on the space of spin interactions have coupling constants

$$
\begin{aligned}
(\pi_{\text{spin}}\Phi_n)(X) &= \sum_{s:D_n+s\supset X} \frac{1}{|D_n|}\,\mathrm{Tr}\left\{\left[\prod_{j\in X}\sigma_j \exp\left(\frac{i}{|D_n|^{1/2}}\,\sigma_j\right)\right]\right.\\
&\quad\times\left.\left[\prod_{j\in\{D_n+s\}\backslash X}\exp\left(\frac{i}{|D_n|^{1/2}}\,\sigma_j\right)\right]\right\}\\
&= \frac{|\{s: D_n+s\supset X\}|}{|D_n|}\left[i\sin\frac{1}{|D_n|^{1/2}}\right]^{|X|}\left[\cos\frac{1}{|D_n|^{1/2}}\right]^{|\{D_n+s\}\backslash X|}
\end{aligned}
\tag{10}
$$

whose norms satisfy

$$
\begin{aligned}
\|\pi_{\text{spin}}\Phi_n\|^{\text{spin}}_{g=1} &\geqslant \frac{1}{|D_n|}\sum_{\substack{Y\ni 0\\ Y\subset D_n}}\left[\sin\frac{1}{|D_n|^{1/2}}\right]^{|Y|}\left[\cos\frac{1}{|D_n|^{1/2}}\right]^{|D_n\backslash Y|}\\
&= \sum_{l=1}^{|D_n|}\binom{|D_n|-1}{l-1}\left[\sin\frac{1}{|D_n|^{1/2}}\right]^{l}\left[\cos\frac{1}{|D_n|^{1/2}}\right]^{|D_n|-l}\\
&= \frac{1}{|D_n|}\left[\sin\frac{1}{|D_n|^{1/2}}\right]\left[2^{1/2}\sin\left(\frac{1}{|D_n|^{1/2}}+\frac{\pi}{4}\right)\right]^{|D_n|-1}\\
&\xrightarrow[n\to\infty]{} \exp[O(|D_n|^{1/2})]\to\infty
\end{aligned}
\tag{11}
$$

Expressions (9) and (11) prove that π_{spin} is unbounded, and *a forteriori* they also show that π_{gas} is unbounded because $\mathscr{T}\pi_{\text{gas}}\Phi_n=\pi_{\text{spin}}\Phi_n$ and $\mathscr{T}$ is a bounded map.

Moreover, these examples prove that

$$\overline{\mathscr{B}^{\text{spin}}_g\backslash\mathscr{B}^{\text{gas}}_g}=\mathscr{B}^{\text{spin}}_g \tag{12}$$

$$\overline{[\mathscr{B}_g]\backslash\mathscr{B}^{\text{spin}}_g}=\mathscr{B}_g \tag{13}$$

and

$$\overline{[\mathscr{B}_g]\backslash\mathscr{B}^{\text{gas}}_g}=\mathscr{B}_g \tag{14}$$

{For instance, the interactions $\Phi_{(N)}=\sum_{n\geqslant N}\Phi_n/|D_n|^2$ are in $\mathscr{B}_g\backslash\mathscr{B}^{\text{spin}}_g$ [by (11)], and

$$\Phi_{(N)}\xrightarrow[N\to\infty]{\|\ \|_g} 0$$

by (9). Similarly, $\Phi^{\text{spin}}_{(N)}=\sum_{n\geqslant N}\Phi^{\text{spin}}_n/|D_n|^2$ are in $\mathscr{B}^{\text{spin}}_g\backslash\mathscr{B}^{\text{gas}}_g$ [by (7)], and

$$\Phi^{\text{spin}}_{(N)}\xrightarrow[N\to\infty]{\|\ \|^{\text{spin}}_g} 0$$

by (6).}

The inequivalence of the different formulations implied by the unboundedness of the maps $\pi_{\rm spin}$, $\pi_{\rm gas}$, and $\mathscr{T}^{-1}$, is seldomly mentioned in the literature. One reason for this is that some general results are already valid in the biggest spaces $\mathscr{B}_g$ and they are inherited by the subspaces $\mathscr{B}_g^{\rm spin}$ and $\mathscr{B}_g^{\rm gas}$. Among these, we cite the existence of the pressure (and therefore of invariant equilibrium states) for $g(t)=1/t$, and the existence of Gibbs states (DLR conditions) for $g(t)=1$ (see, for example, ref. 6). The situation is, however, more delicate for results regarding the analyticity of the free energy and correlation functions. Such analyticity (with respect to a finite number of parameters of the interaction, such as temperature or density), is physically interpreted as the absence of phase transitions. Initially, it was proven[3] that in the smaller lattice-gas space $\mathscr{B}_{g=1}^{\rm gas}$, such analyticity holds for interactions $\Phi^{\rm gas}$ with $\|\Phi^{\rm gas}\|_{g=1}^{\rm gas}$ small enough. In view of (12) this result could, in principle, fail to be true if the larger spin space $\mathscr{B}_{g=1}^{\rm spin}$ were considered. A small change in the parameters of an interaction could produce an interaction outside $\mathscr{B}_{g=1}^{\rm gas}$ where analyticity is questionable. However, such a generalization was later proved in ref. 5: within $\mathscr{B}_{g=1}^{\rm spin}$ small interactions are analytic with respect to finitely many parameters.

On the other hand, analyticity in the largest spaces $\mathscr{B}_g$ was proven only for the more restrictive norms defined by $g(t)=e^{\lambda t}$ with $\lambda \gneqq 0$[2,5] (in fact, for systems with a very general compact one-particle space; and also for the quantum case[11–15]). The result of ref. 1 shows that this cannot be improved: if g grows more slowly than an exponential, there is no analyticity result in $\mathscr{B}_g$.

It seems worthwhile to stress this different analytic behavior with respect to the norms $\|\cdot\|_{g=1}^{\rm spin}$, $\|\cdot\|_{g=1}^{\rm gas}$, and $\|\cdot\|_{g=1}$. The possibility of such a difference has not been remarked upon in the literature; in fact, the behavior has been (implicitly or explicitly) assumed to be the same.[5,7–10] (In fact, at first, we, too, suspected that ref. 1 contradicted ref. 3 or ref. 5.) The reason for this different behavior is contained in (13): the lack of analyticity in a neighborhood—in $\mathscr{B}_g$—of a given "small" interaction in $\mathscr{B}_{g=1}^{\rm spin}$ (e.g., zero!) is due to interactions not in $\mathscr{B}_{g=1}^{\rm spin}$. In fact, the process devised in ref. 1 to construct a (complex) interaction $\Phi\in\mathscr{B}_{g=1}$ with $\|\Phi\|_{g=1}<2\varepsilon$ for which the pressure in not analytic (in fact not even defined) starts with an interaction of the form

$$\Phi_X^{(2)}(\sigma)=\begin{cases} i\varepsilon\sigma_j & \text{if } X=\{j\} \\ -\ln\left[1-(\cos\varepsilon)^{|\Lambda|}\exp(-i\varepsilon\sum_{j\in X}\sigma_j\right] & \text{for } X \text{ translate of } \Lambda \\ 0 & \text{otherwise}\end{cases} \tag{15}$$

where $\Lambda \subset \mathbb{Z}^d$ is big enough so that

$$|\Lambda|(\cos \varepsilon)^{|\Lambda|} < \varepsilon/4 \tag{16}$$

[such an interaction satisfies $Z_\Lambda(\Phi) = 0$, where Z_Λ is the partition function in Λ with free boundary conditions]. The interaction (15) is very similar to our example (8) and it is not hard to see that $\|T^{-1}\Phi^{(2)}\|_{g=1}^{\text{spin}} \to_{\varepsilon \to 0} \infty$, as it should do in view of the results of refs. 3–5.

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